Minimal relaxation law for compaction of tapped granular matter

Stefan J. Linz and Annekathrin Döhle
Theoretische Physik I, Institut für Physik, Universität Augsburg, D-86135 Augsburg, Germany
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Granular systems can compact under the influence of sufficiently strong, successive tapping. Recent experimental investigations show that the packing fraction obeys a very slow relaxation to a final, dense packing fraction that is basically proportional to the inverse of the logarithm of the tap number or time. We provide a simple macromechanical argument that explains this inverse logarithmic relaxation in time in all functional details. By considering the asymptotic limits of the resulting relaxation law, we show that the relaxational dynamics of the compaction process can be interpreted as a combination of a biased void diffusion for short times and a collective reorganization for large times. [S1063-651X(99)02611-2]

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I. INTRODUCTION

It is a part of our childhood experience that dry sand in a pail can be densified by tapping the pail on the ground if the tapping strength is great enough. This property, the densification of granular matter by reducing the void volume between the grains, is commonly called compaction. Granular matter such as sand and powder consists of large collections of dry massive macroscopically extended particles that interact only via repulsive forces. It is the absence of attractive forces between the grains that allows for loosening and reorganization of the grains under the influence of external forces such as tapping or shaking. This feature shows very evidently the distinction of granular matter from ordinary solids and Newtonian fluids.

Despite its obvious technological and practical importance, detailed quantitative experimental studies on the compaction behavior of granular matter have begun only recently (for an overview, cf. Refs. [1–3]). In a series of seminal works, the group of Jaeger and Nagel [4–6] have investigated the ensemble-averaged settling of monodisperse granular particles in a long vertical tube into more compacted states by applying periodic vertical tapping with a constant tapping intensity Γ. Since the tapping intensity Γ is defined by the ratio of the peak intensity of the tap and the gravitational acceleration, one expects that only for $\Gamma > 1$ do the grains experience enough upward acceleration to loosen and to reorganize.

In their experiments performed for tapping intensities $\Gamma > 1.4$, Knight *et al.* [4] were able to fit their data for the increase of the packing fraction ρ with time or tap number t to a surprisingly simple functional form, given explicitly by

$$\rho(t) = \rho_\infty - \frac{\rho_\infty - \rho_0}{1 + B \ln(1 + t/\tau)}. \tag{1}$$

Here, B and τ are coefficients that strongly depend on the tapping intensity Γ, but not on the time t. $\rho_0 = \rho(t=0)$ denotes the initial packing fraction of a loosely packed state, and $\rho_\infty = \rho(t=\infty)$ denotes the final packing fraction that also depends on Γ. This comparatively slow relaxation to the final packing fraction did not seem to corroborate previous numerical studies [7,8], in particular the algebraic relaxation of the packing fraction with time suggested by Hong *et al.* [8].

In the wake of this study, several apparently rather distinct theoretical explanations based on microscopic [9,10], mesoscopic [11–13], or macroscopic [14,15,17] ideas have been developed to understand the origin of the compaction formula, Eq. (1). Although Eq. (1) is clearly the result of a highly complicated micromechanical reorganization of the collective of grains, is there a simple macromechanical mechanism that explains the functional details of Eq. (1)? The answer to this question seems still to be unresolved and is the main topic of this paper.

The purpose of our paper is fourfold. First, we show that the functional form of the fit formula, Eq. (1), can be derived from one simple mean-field argument about the dynamical changes of the void volume under tapping. Second, by considering the asymptotic limits of the resulting relaxation law, we interpret our mean-field argument as a smooth interpolation between two known mechanisms, namely the biased void diffusion [8] and collective reorganization [14,15]. Third, we show that a previously suggested stroboscopic model [17] is also consistent with the proposed dynamical mechanism. Fourth, we propose a simple picture of the solid-compactible phase transition that is based on our mean-field argument.

II. MACROSCOPIC PICTURE OF THE COMPACTION PROCESS

On a purely macroscopic level, the total volume V_t of a granular system consisting of a fixed, large number of particles is made up of two components: (i) the grain space V_g representing the part of the total volume V_t that is occupied by the grains, and (ii) the void space V_v characterizing the empty intergranular part of the total volume V_t. The void space V_v, again, can be divided into two components. (i) The *removable* void space V_r. This part characterizes the contribution to the void volume that can be effectively extracted from the granular system by tapping. (ii) The *irremovable* void space V_i. This part consists of the contribution to the void volume that survives the compaction process and results from the fact that a compacted granular system cannot

be totally space filling. The grain space V_g and the irremovable void space V_i do not depend on the time or tap number during the compaction process, whereas the removable void space $V_r(t)$ diminishes under tapping from initial removable void volume $V_r(0)$ to the minimal removable void volume in the long time limit, $V_r(\infty)=0$. Therefore, the time dependence of the total volume V_t during the compaction process is determined by

$$V_t(t)=V_g+V_v(t)=V_g+V_i+V_r(t) \tag{2}$$

with $V_r(t)$ being the only time-dependent contribution. The packing fraction is defined by the ratio of the grain space and the total volume,

$$\rho(t)=V_g/V_t(t), \tag{3}$$

whereas the effective reduced void volume is determined by the ratio of the removable void volume and the total volume,

$$\chi(t)=V_r(t)/V_t(t). \tag{4}$$

Obviously, $\rho(t)$ and $\chi(t)$ are related by $V_r(t)\rho(t)=V_g\chi(t)$. For the following, it is convenient to introduce the *inverse (effective) void ratio* given by

$$X(t)=\chi(0)/\chi(t), \tag{5}$$

where $X(t)$ can vary between its initial value $X(0)=1$ and the final value $X(\infty)=\infty$.

III. MINIMAL RELAXATIONAL LAW

Here, we want to show that the inverse logarithmic relaxation of the packing fraction, Eq. (1), found by Knight *et al.* [4] can be derived from one *single* mean-field argument about the dynamics of the (inverse) void ratio with time or, respectively, tapping. Using the aforementioned macroscopic picture, our main statement about the time evolution of the compaction process driven by periodic tapping can be formulated as follows:

If the time rate of change of the inverse void ratio dX/dt slows down exponentially with increasing inverse void ratio X, the packing fraction obeys the functional form of Eq. (1).

To verify this statement, we write the basic assumption explicitly in the form

$$\dot{X}(t)=m_1\exp[-m_2X(t)] \tag{6}$$

with m_1 and m_2 being positive constants with respect to time. In general, however, they will depend on the tapping intensity Γ and the micromechanical properties of the granular system. A straightforward integration of Eq. (6) yields

$$\int_0^t dt=\frac{1}{m_1}\int_1^{X(t)} dX\exp(m_2X)$$

$$=-\frac{e^{m_2}}{m_1m_2}(1-\exp\{-m_2[1-X(t)]\}), \tag{7}$$

where the initial condition $X(0)=1$ has been inserted. Solving the resulting implicit equation for $X(t)$, one immediately infers that the inverse void ratio evolves according to

$$X(t)=1+\frac{1}{m_2}\ln[1+m_1m_2e^{-m_2}t]. \tag{8}$$

Using the fact that the inverse void ratio $X(t)$ and the packing fraction $\rho(t)$ are related by (cf. the Appendix)

$$\rho(t)=\rho_\infty+(\rho_0-\rho_\infty)X^{-1}(t) \tag{9}$$

and identifying the constants entering in Eq. (1) and Eq. (8),

$$1/B=m_2 \quad \text{and} \quad 1/\tau=m_1m_2e^{-m_2}, \tag{10}$$

one immediately recovers the functional form of Eq. (1).

As a consequence, the *minimal relaxation law* for compaction driven by periodic tapping expressed in terms of the coefficients B and τ introduced by Knight *et al.* [4] reads explicitly

$$\tau\dot{X}=B\exp[(1-X)/B] \tag{11}$$

and is one of the central results of this paper. The limits $\tau\to\infty$ or $B\to0$ of Eq. (11) represent in a natural way the solid matter limit where no compaction can take place at all, $X(t)=X(0)$ for all t.

Next, it is important to note that the coefficients B and τ are not independent as far as their functional dependence on the tapping intensity is concerned. Based on the experimental results [4], it has been suggested in Ref. [17] that the coefficients B and τ obey the relation $B=\kappa\tau$, with κ being independent of the tapping intensity Γ and given by $\kappa\approx1/18$ for the experimental setup in Ref. [4]. Therefore, we can recast Eq. (11) in the form

$$\dot{X}=\kappa\exp[(1-X)/\kappa\tau], \tag{12}$$

where the characteristic decay time τ is now the only coefficient that depends on the tapping intensity Γ. Typically, τ varies from values of the order 10^5 for tapping intensities $1.4<\Gamma<2$ to values of about 2 for $\Gamma>3$ [4].

IV. CONNECTION WITH ALTERNATIVE MODELS

To relate Eq. (6) to previously suggested mesoscopic arguments [8,14,15] for relaxational compaction, it is convenient to introduce the *reduced packing fraction*

$$A(t)=\frac{\rho(t)-\rho_\infty}{\rho_0-\rho_\infty}. \tag{13}$$

This quantity can vary between $A(0)=1$ and $A(\infty)=0$ during the compaction process. Noting that (i) the reduced packing fraction $A(t)$ is related to the inverse reduced void ratio $X(t)$ by $A(t)=X^{-1}(t)$ (cf. the Appendix) and (ii) $\dot{A}(t)=-X^{-2}(t)\dot{X}(t)=-A^2(t)\dot{X}(t)$ holds, the minimal relaxation law, Eq. (12), expressed in terms of $A(t)$ reads

$$\dot{A}=-\kappa A^2\exp[-(1-A)/\kappa\tau A]. \tag{14}$$

Using Eq. (9), one directly infers that Eq. (14) possesses the solution $A(t)=1/[1+\kappa\tau\ln(1+t/\tau)]$. Next we discuss the asymptotic limits of Eq. (14) for small and large times or tap numbers.

Since the compaction process starts with $A(0)=1$, the *short time asymptotics* of Eq. (14) can be obtained by expanding the exponential term in Eq. (14) about $A=1$. This yields

$$\dot{A}=-\kappa A^2[1-(1-A)/\kappa\tau A+\text{higher-order terms}]. \tag{15}$$

As a consequence, Eq. (14) reduces to $\dot{A}=-\kappa A^2$ if $1+\kappa\tau \gg 1/A$ or, equivalently, $t \ll 2\tau$. This directly leads to an algebraic decay $A(t)=1/(1+\kappa t)$ and, therefore, to a relaxation of the packing fraction $\rho(t)\propto 1/t$ for short times, $t \ll 2\tau$. Based on the idea of biased void diffusion as a result of tapping granular matter of height $z(t)$ in a vertical tube, Hong *et al.* [8] found an algebraic relaxation of $z(t)$ to a finite height $z_c=z(\infty)$ being proportional to the time or tap number t and saturating after a finite time t_c. Since the mass of the granular system is conserved and, therefore, the height $z(t)$ is inversely proportional to the packing fraction $\rho(t)$, Hong *et al.*'s result [8] agrees with our findings for short enough times $t \ll 2\tau$.

To understand the long time asymptotics of Eq. (14), it is convenient to recast Eq. (14) in the form

$$\dot{A}=-\kappa\exp(1/\kappa\tau)\exp[-(1/\kappa\tau A)(1-\kappa\tau 2A\ln A)]. \tag{16}$$

The reduced packing fraction $A(t)$ becomes very small in the long time limit. This implies that (i) $A\ln A$ approaches zero as $t\to\infty$ and (ii) Eq. (14) is asymptotically equivalent to $\dot{A}=-\kappa\exp[-(1-A)/\kappa\tau A]$ for long times or tap numbers. (This argument can be generalized. Any relaxational dynamics of the form $\dot{A}=Q_N(A)$, with $Q_N(A)=-f_1A^N\exp[-(1-A)/f_2A]$ with f_1 and f_2 being positive coefficients and N being an integer and positive, can be rewritten in the form $\dot{A}=Q_N(A)=-f_1\exp(1/f_2)\exp[-(1/f_2A)(1-f_2NA\ln A)]$. Since the term $A\ln A$ approaches zero as A approaches zero, $Q_N(A)$ approaches asymptotically $Q_0(A)$ in the limit $A\to 0$. Therefore, any relaxational dynamics of the form $\dot{A}=Q_N(A)$ will eventually show a relaxation proportional to $1/\ln t$.) The latter relaxation law, however, is exactly the functional form that results from the collective reorganization argument put forward by Ben Naim *et al.* [14] on the basis of the one-dimensional parking lot model [14] and by Boutreux and de Gennes [15] on the basis of a free volume argument. It is based on the idea [3] that an increase of the density of an already packed system requires a collective reorganization of a large part of the system and is exponentially costly for particles that can move independently and randomly.

To summarize, our macromechanical argument (6) or, equivalently, the relaxation law for the reduced packing fraction, Eq. (14), *unifies in a natural way* two mesoscopic mechanisms for granular compaction that are dominant on different time scales of the compaction process and are *both* needed to explain the experimental result, Eq. (1). For short times $t \ll \tau$, biased void diffusion [8] with an algebraic relaxation $\rho(t)-\rho(\infty)\propto 1/t$ dominates granular compaction, whereas the collective reorganization with a relaxation $\rho(t)\propto 1/\ln t$ dominates for long times $t \gg \tau$. The crossover of these effects happens when t reaches the size of τ. Due to the aforementioned strong dependence of the relaxation time τ on the tapping amplitude Γ [4], the range of dominant void diffusion can vary over several decades. In particular, the biased void diffusion seems to be an essential ingredient for the understanding of compaction for comparatively low tapping amplitudes, $1<\Gamma<2$.

V. DERIVATION OF THE RELAXATIONAL LAW FROM THE STROBOSCOPIC MODEL

Here, we show that the minimal relaxation law for compaction, Eq. (11), can also be interpreted as the coarse-grained time-continuous limit of the stroboscopic model for compaction proposed previously in Ref. [17]. This model is also supported by the cluster dynamical approach of Gavrilov [12]. Based on a comparatively crude dynamical modeling of the response of the granular system to a single tap, it has been suggested in Ref. [17] that the relaxational compaction process is governed by a stroboscopic nonlinear map or difference equation for the *compaction ratio*, i.e., the appropriately rescaled packing fraction after each tap $n=1,2,3,\ldots$,

$$\alpha_n=\frac{\rho_n-\rho_\infty}{\rho_0-\rho_\infty}. \tag{17}$$

The difference equation governing the stroboscopic dynamics of the compaction process is given by [17]

$$\alpha_n=f(\alpha_{n-1},n)=\frac{\alpha_{n-1}}{1+h_n\alpha_{n-1}}, \tag{18}$$

where h_n denotes a memory term that depends on the tap number n via

$$h_n=C/(1+n/\nu) \tag{19}$$

and labels how far the compaction process has progressed already. In Eq. (19), C and ν are positive coefficients that are independent of time, but dependent on the tap intensity. Using the initial condition $\alpha_0=1$, the exact solution of Eqs. (18) and (19) can be expressed in terms of digamma functions and can be considered as the iterated equivalent [17] of Knight *et al.*'s formula, Eq. (1).

A time-continuous coarse-grained relaxational law that interpolates the discrete variations of the compaction ratio resulting from Eq. (18) can be achieved by substituting

$$\alpha_{n-1}\leftrightarrow A(t), \tag{20}$$

$$\alpha_n-\alpha_{n-1}\leftrightarrow\dot{A}(t), \tag{21}$$

$$h_n\leftrightarrow h(t)=C/(1+t/\nu), \tag{22}$$

where $A(t)$ is merely but the reduced packing fraction introduced in Eq. (13). The resulting time-continuous relaxational equation reads

$$\dot{A}(t)=\left(\frac{1}{1+h(t)A(t)}-1\right)A(t). \tag{23}$$

Since the coefficient C is typically of the order 10^{-2} [17] and $A(t) \leqslant 1$ for the whole compaction process, one infers that the product $h(t)A(t)$ in the denominator of Eq. (23) is small and even diminishes as time or compaction proceeds. Therefore, we can expand the denominator in Eq. (23) in powers of $h(t)A(t)$ and obtain in the leading-order approximation

$$\dot{A}(t) = -h(t)A^2(t). \tag{24}$$

Solving Eq. (24) with $h(t)$ given by Eq. (22) and the initial condition $A(0)=1$ yields $A(t)=1/[1+C\nu \ln(1+t/\nu)]$. Therefore, it leads again directly to Knight *et al.*'s fit formula, Eq. (1). To relate the memory term $h(t)$ that depends so far on the tap number to the dynamics of the compaction process, i.e., to express $h(t)$ as a function of A, one can take advantage of the relation $C\nu \ln(1+t/\nu) = 1/A(t) - 1$ that follows from a combination of the aforementioned solution $A(t)$ and Eq. (22). This implies that Eq. (24) can also be written in the equivalent form

$$\dot{A} = -h(A)A^2, \tag{25}$$

$$h(A) = C \exp[-(1-A)/C\nu A]. \tag{26}$$

With the trivial renaming $C=\kappa$ and $\nu=\tau$, Eq. (25) in combination with Eq. (26) is obviously equivalent to Eq. (14) and, therefore, also equivalent to the basic relaxation law, Eq. (7). Moreover, one infers that the exponential factor in Eq. (26) directly results from the memory term h_n or $h(t)$ in the stroboscopic or time-continuous description, respectively.

VI. SOLID-COMPACTIBLE PHASE TRANSITION

As pointed out above, a granular system behaves like a solid for tapping intensities $\Gamma < 1$ since in that case the grains cannot lift off or decompact and subsequently recompact as a response to a tap. For $\Gamma > 1$, the granular system is compactible due to tapping and its relaxational compaction dynamics is governed by Eq. (12) with only *one* coefficient that depends on the tap intensity, namely the relaxation time τ. Introducing $\Theta(t) = \rho_0 - \rho(t)$ as an appropriate order parameter, $\Theta(t)$ is zero for $\Gamma < 1$ (the solid phase) and relaxes to the nonzero positive value $\rho_0 - \rho(\infty)$ (being dependent on Γ) in the long time limit if $\Gamma > 1$ (compactible phase). The relaxational dynamics is governed by the relaxation time τ, which functionally depends on the distance from the phase transition point $\Gamma = 1$, i.e., $\tau = \tau(\Gamma - 1)$. The functional dependence of τ on $\Gamma - 1$ should slow down in a critical way, $\tau(\Gamma - 1 \to 0) \to \infty$, and decay rapidly for Γ larger than unity since a reorganization of the grains is easier if the loosening of the granular system as an initial response to a tap is stronger. Although the general feature of a relaxation time τ that strongly increases when approaching $\Gamma = 1$ is supported by the experiments of Knight *et al.* [4], the existing data do not corroborate a divergent behavior of τ as Γ approaches unity. This is an interesting open problem for further experimental investigations.

VII. CONCLUSION

Despite the fact that compaction of granular matter driven by periodic tapping is micromechanically highly complex, we have reported a rather simple macromechanical mechanism for the inverse effective void ratio that correctly reproduces the experimental findings of Knight *et al.* [4], Eq. (1). By considering the asymptotic limits of our mean-field model, we were able to identify Eq. (14) as a smooth interpolation of biased void diffusion [8] and collective reorganization [14,15]. Both macroscopic mechanisms for the dynamics of the packing fraction are rather robust in the sense that the micromechanical details of the granular material (e.g., the shape of the particles) are of minor relevance. This might be the underlying reason why recent numerical simulations of Tetris-like models with a variety of different particle shapes [16] (e.g., ball-shaped, T-shaped, L-shaped) also reproduce the experimental results [4].

As far as generalizations of our mean-field model are concerned, the application of previously successful ideas about the extension of mean-field models to Langevin-type equations in the context of avalanches in granular systems [18] seems to be promising for the understanding of the spectral behavior of compaction processes [5]. On the other hand, it remains an open question for future research whether the dynamical systems approach being used in this study can also be extended to understand the annealing behavior of the packing fraction under dynamical increase and decrease of the tap intensity [6].

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APPENDIX

Here, we give the details leading to the relation (9). According to Eq. (3), the packing fraction is given by

$$\rho(t) = \frac{V_g}{V_t(t)} = \frac{V_g}{V_g + V_i + V_r(t)}, \tag{A1}$$

which possesses the saturation limit

$$\rho(\infty) = \frac{V_g}{V_g + V_i}. \tag{A2}$$

From this, we obtain by straightforward algebra

$$\rho(t) - \rho_\infty = -\frac{V_g V_r(t)}{[V_g + V_i + V_r(t)][V_g + V_i]} \tag{A3}$$

and finally arrive at

$$\frac{\rho(t) - \rho_\infty}{\rho_0 - \rho_\infty} = \frac{V_r(t)[V_g + V_i + V_r(0)]}{[V_g + V_i + V_r(t)]V_r(0)} = \frac{\chi(t)}{\chi(0)} = \frac{1}{X(t)}. \tag{A4}$$

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